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ON FRACTIONAL s^m FACTORIAL DESIGNS

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Abstract. Statistical structure of the fraction of symmetrical s^m factorial designs is investigated in some detail. In this paper, we show that the information matrix of a fractional s^m factorial design is determined completely by its characteristic vector. We also give an explicit expression of the elements of the infomation matrix of the design derived from s-symbol balanced arrays in terms of its indices.

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§1. Introduction

In his pioneering work, Taguchi [12] contributed to the extensive use of orthogonal fraction of 2^m factorial designs obtained by assigning the factors to the appropriately selected column of saturated orthogonal arrays, or 'orthogonal tables'. At that time, Box and Hunter [2, 3] investigated the structure of the orthogonal fraction of 2^m factorial designs at length. Orthogonal fractions, however, require much more than desirable number of assemblies or treatment combinations. Sometimes, it becomes infeasible if the higher power of resolution is expected. Balanced fractions based on the concept of balanced arrays were investigated among others by Srivastava [10], Srivastava and Chopra [11] and Yamamoto, Shirakura and Kuwada [13, 14]. These investigations were concerned with the structure of balanced fractional 2^m factorial designs. In his work, Kuwada [5, 6, 7] contributed in the analysis of balanced fractional 3^m factorial designs. His work was extended to the balanced fractional s^m factorial cases by Kuwada and Nishii [8, 9].

In this paper, the structure of the fraction of symmetrical s^m factorial designs will be investigated in some detail. We shall show that the information matrix of a fractional s^m factorial design is determined completely by its characteristic vector. We shall also give an explicit expression of the elements of the infomation matrix of the design derived from s-symbol balanced arrays in terms of its indices.

§2. s^m factorial designs

Consider an s^m factorial experiment with m factors $F(p),\ p \in \Omega = \{1, 2, \cdots, m\}$, each at levels $i_p \in S = \{0, 1, \cdots, s-1\}$. Let $y(\boldsymbol{j}')$ and $\eta(\boldsymbol{j}')$ be the observation and its expectation of an assembly or a treatment combination $\boldsymbol{j}' = (j_1, j_2, \cdots, j_m)$ expressed by an s-ary row vector, respectively.

Let $\boldsymbol{Z}$ be the arrangement of all possible s^m s-ary row vectors in their lexicographic order and let $\boldsymbol{y}(\boldsymbol{Z})$ and $\boldsymbol{\eta}(\boldsymbol{Z})$ be the observation and the expectation of s^m dimensional vector of corresponding assemblies on the complete s^m factorial design, respectively. The vector of factorial effects $\Theta(\boldsymbol{Z})$ and its components $\theta(\boldsymbol{i}')$ for $\boldsymbol{i}' = (i_1, i_2, \cdots, i_m)$ based on the orthogonal decomposition of effects between levels may be defined as follows:

$$\Theta(\boldsymbol{Z}) = \frac{1}{s^m} D'_{(m)} \boldsymbol{\eta}(\boldsymbol{Z}), \tag{2.1}$$

where $D_{(m)} = D \otimes D \otimes \cdots \otimes D$ denotes the m-times Kronecker products of an $s \times s$ matrix $D = [\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{s-1}]$. Those s columns $\boldsymbol{d}'_i = (d_{0i} d_{1i} \cdots d_{s-1i})$ of D with $\boldsymbol{d}'_0 = (11 \cdots 1)$ satisfy the orthogonality condition $\boldsymbol{d}'_i \boldsymbol{d}_k = s\delta_{ik}$ with Kronecker δ_{ik} for every i and k in S.

We may note that the definition of factorial effects here is designed to keep *homoscedasticity* among their BLUE's obtained under the complete s^m factorial design.

Solving (2.1), we have

$$\boldsymbol{\eta}(\boldsymbol{Z}) = D_{(m)} \Theta(\boldsymbol{Z}).$$

Let $U^x = \{p | i_p = x\}$ be a subset of Ω in which the argument i_p of $\theta(\boldsymbol{i}')$ is equal to x for every $x \in S$. Then the factorial effect $\theta(\boldsymbol{i}')$ can be expressed as $\theta(U^0 U^1 \cdots U^{s-1})$ or alternatively as $\theta(U^1 U^2 \cdots U^{s-1})$ by indicating $s-1$ subsets $U^x,\ x \in S' = S - \{0\}$, since $d_{j0} = 1$ for every j. Some of those U^x, however, may be omitted if they are null.

If $| \cup_{x \in S'} U^x | = 0$, the parameter or factorial effect $\theta(0, 0, \cdots, 0, 0)$ is called the general mean and is denoted alternatively by $\theta(\phi)$. If $| \cup_{x \in S'} U^x | = 1$ and $U^{i_p} = \{p\}$ for a nonzero i_p, then the parameter $\theta(0, 0, \cdots, i_p, \cdots, 0)$ is called the i_pth order main effect of the factor $F(p)$ and is denoted alternatively by

$\theta(p^{i_p})$. If $|\cup_{x\in S'} U^x| = 2$ and $\cup_{x\in S'} U^x = \{p, q\}$, then the parameter $\theta(\boldsymbol{i}')$ having two nonzero i_p and i_q is called the $i_p \times i_q$ order two-factor interaction of the factors $F(p)$ and $F(q)$. Such a two-factor interaction can be denoted alternatively by $\theta(p^{i_p} q^{i_q})$. In general, if $|\cup_{x\in S'} U^x| = k$, then the parameter $\theta(\boldsymbol{i}')$ having k nonzero arguments with respect to k factors is called the k-factor interaction and is denoted as compact as possible by indicating the sets of non-null arguments.

Let T be a fraction of s^m factorial design with m factors and n assemblies whose αth row is $\boldsymbol{j}^{(\alpha)\prime} = (j_1^{(\alpha)}, j_2^{(\alpha)}, \cdots, j_m^{(\alpha)})$; $j_p^{(\alpha)} \in S,\ p \in \Omega,\ \alpha = 1, 2, \cdots, n$; and suppose $\boldsymbol{y}(T)$ is the corresponding vector of n observations. Then, $\boldsymbol{y}(T)$ can be expressed as

$$\boldsymbol{y}(T) = E(T)\Theta + \boldsymbol{e}(T), \tag{2.2}$$

where Θ is the parameter vector obtained by rearranging $\Theta(\boldsymbol{Z})$ in a natural order of the number of factors and the order of the levels of factors concerned, $E(T)$ is the design matrix of size $n \times s^m$ and $\boldsymbol{e}(T)$ is the error vector with usual assumption that the components are distributed independently with $(0, \sigma^2)$.

Since $d_{j0} = 1$ for every j, the column vector of the design matrix $E(T)$ corresponding to the factorial effect $\theta(U^1 U^2 \cdots U^{s-1})$ is expressed as:

$$\begin{aligned} &\boldsymbol{L}(\theta(U^1U^2\cdots U^{s-1})) \\ &= \Big(\prod_{x\in S'}\prod_{p_x\in U^x} d_{j_{p_x}^{(1)}x}, \prod_{x\in S'}\prod_{p_x\in U^x} d_{j_{p_x}^{(2)}x}, \cdots, \prod_{x\in S'}\prod_{p_x\in U^x} d_{j_{p_x}^{(n)}x}\Big)'. \end{aligned} \tag{2.3}$$

Definition 2.1. For a fractional s^m factorial design T, the vector $\boldsymbol{L}(\theta(U^1U^2 \cdots U^{s-1}))$ is called the *loading vector* of a factorial effect $\theta(U^1U^2\cdots U^{s-1})$.

Using loading vectors of $m(s-1)$ main effects, every loading vector can be obtained by enumerating the Schur product $(*)$ of a certain number of related loading vectors for main effects as is given in (2.3). For example, we have $\boldsymbol{L}(\theta(p^{i_p}q^{i_q})) = \boldsymbol{L}(\theta(p^{i_p})) * \boldsymbol{L}(\theta(q^{i_q}))$.

Let $S_p[\boldsymbol{x}]$ be the *spur* of a vector $\boldsymbol{x}$ being defined by the sum of its components.

Definition 2.2. The spur $S_p[\boldsymbol{L}(\theta(U^1U^2\cdots U^{s-1}))]$ of the loading vector of an effect $\theta(U^1U^2\cdots U^{s-1})$ given by

$$\gamma(\theta(U^1U^2\cdots U^{s-1})) = \sum_{\alpha=1}^{n}\prod_{x\in S'}\prod_{p_x\in U^x} d_{j_{p_x}^{(\alpha)}x}$$

is called the *loading coefficient* of the factorial effect $\theta(U^1U^2\cdots U^{s-1})$ of the design T.

In particular, $\gamma(\theta(\phi)) = n$, $\gamma(\theta(p^{i_p})) = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)} i_p}$ and $\gamma(\theta(p^{i_p} q^{i_q})) = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)} i_p} d_{j_q^{(\alpha)} i_q}$ for $p,\ q \neq p \in \Omega$ and $i_p,\ i_q \in S'$.

The normal equation for estimating Θ is given by

$$M(T)\Theta = E(T)'\boldsymbol{y}(T), \tag{2.4}$$

where $M(T) = E(T)'E(T)$ is the information matrix of a design T.

The following is a lemma due to Kuwada and Nishii [8].

Lemma 2.1. *Every Schur product of two column vectors $\boldsymbol{d}_i$ and $\boldsymbol{d}_k$ of the matrix D is given by a linear combination of $\boldsymbol{d}_\ell$ as follows:*

$$\boldsymbol{d}_i * \boldsymbol{d}_k = \sum_{\ell=0}^{s-1} c_{ik}^{\ell} \boldsymbol{d}_\ell, \text{ or } d_{ji} d_{jk} = \sum_{\ell=0}^{s-1} c_{ik}^{\ell} d_{j\ell} \text{ holds for every } j,$$

where the constant coefficients satisfy $c_{ik}^{\ell} = c_{ki}^{\ell}$ and are given by $c_{ik}^{\ell} = \boldsymbol{d}_\ell'(\boldsymbol{d}_i * \boldsymbol{d}_k)/s$. In particular, $c_{ik}^{0} = \delta_{ik}$.

Using Lemma 2.1, we have:

Theorem 2.2. *The element $\varepsilon(\theta(U^1U^2\cdots U^{s-1}), \theta(V^1V^2\cdots V^{s-1}))$ of the information matrix $M(T)$ of a fractional s^m factorial design T corresponding to the $\theta(U^1U^2\cdots U^{s-1})$ row and $\theta(V^1V^2\cdots V^{s-1})$ column is given by*

$$\begin{aligned} &\varepsilon(\theta(U^1U^2\cdots U^{s-1}), \theta(V^1V^2\cdots V^{s-1})) \\ &\quad = \sum_{\alpha=1}^{n} \prod_{x,y\in S} \prod_{p_{xy}\in K^{xy}} \Big(\sum_{\ell=0}^{s-1} c_{xy}^{\ell} d_{j_{p_{xy}}^{(\alpha)} \ell}\Big), \end{aligned} \tag{2.5}$$

where $K^{xy} = U^x \cap V^y$ for every pair of $x,\ y \in S$.

Definition 2.3. The first row $\boldsymbol{\Gamma}(T)$ of the information matrix $M(T)$ which is composed of all loading coefficients $\gamma(\theta(U^1U^2\cdots U^{s-1}))$'s arranged in a natural order of $\theta(U^1U^2\cdots U^{s-1})$'s is called the *characteristic vector* of the design T.

Theorem 2.3. *The information matrix $M(T)$ of the design T is completely determined by its characteristic vector $\boldsymbol{\Gamma}(T)$.*

Proof. The formula (2.5) shows that every component of $M(T)$ is a linear combination of the terms each composed of the sum of the product of at most m $d_{j_p^{(\alpha)} i_p}$'s with respect to α, i.e., a loading coefficient. □

The first member of the normal equation (2.4) is given by

$$n\theta(\phi) + \sum_{k=1}^{m} \sum_{|\cup_{r=1}^{s-1} V^r|=k} \gamma(\theta(V^1V^2\cdots V^{s-1}))\theta(V^1V^2\cdots V^{s-1}) = \boldsymbol{L}(\theta(\phi))'\boldsymbol{y}(T). \tag{2.6}$$

In some sense, the left hand member of the equation (2.6) may be called the *defining formula* of the fractional s^m factorial design T. This is an extension of the *defining relation* introduced by Box and Hunter [2, 3] in the case of fractional 2^m factorial designs.

The member corresponding to an effect $\theta(U^1U^2\cdots U^{s-1})$ is given by

$$\sum_{k=0}^{m} \sum_{|\cup_{r=1}^{s-1} V^r|=k} \varepsilon(\theta(U^1U^2\cdots U^{s-1}), \theta(V^1V^2\cdots V^{s-1}))\theta(V^1V^2\cdots V^{s-1}) = \boldsymbol{L}(\theta(U^1U^2\cdots U^{s-1}))'\boldsymbol{y}(T). \tag{2.7}$$

Those left hand member of (2.7) may be called the *derived formulas* of the design.

§3. Designs derived from s-symbol orthogonal arrays and balanced arrays

Let T be a fractional s^m factorial design composed of n assemblies $\boldsymbol{j}^{(\alpha)\prime}$, $\alpha = 1, 2, \cdots, n$, and consider the characteristic vector $\boldsymbol{\Gamma}(T)$ of the design, that is the first row vector of its information matrix $M(T)$.

Consider a subarray T_{Ω_1} composed of the t columns of T indexed by a t-subset $\Omega_1 = \{p_1, p_2, \cdots, p_t\}$ of Ω and let $\lambda(p_1^{j_{p_1}} p_2^{j_{p_2}} \cdots p_t^{j_{p_t}})$ be the frequency of occurrence of a row $(j_{p_1} j_{p_2} \cdots j_{p_t})$ in the subarray. Consider every element $\gamma(\theta(U^1U^2\cdots U^{s-1}))$ of $\boldsymbol{\Gamma}(T)$ whose arguments satisfy $\cup_{x\in S'} U^x \subset \Omega_1$. Since $d_{j0} = 1$ for every j, $\gamma(\theta(U^1U^2\cdots U^{s-1}))$ may be denoted alternatively as $\gamma(\theta(p_1^{i_{p_1}} p_2^{i_{p_2}} \cdots p_t^{i_{p_t}}))$ by the connection $U^x = \{p_k | i_{p_k} = x\}$ for $x \in S'$ and $U^0 = \Omega_1 - \cup_{x\in S'} U^x = \{p_k | i_{p_k} = 0\}$.

Let $\boldsymbol{\gamma}_{\Omega_1}$ and $\boldsymbol{\lambda}_{\Omega_1}$ be two column vectors obtained by arranging those γ's and λ's in the lexicographic order of $(i_{p_1} i_{p_2} \cdots i_{p_t})$ and $(j_{p_1} j_{p_2} \cdots j_{p_t})$, respectively. Then, since,

$$\begin{aligned}\gamma(\theta(p_1^{i_{p_1}} p_2^{i_{p_2}} \cdots p_t^{i_{p_t}})) &= \sum_{\alpha=1}^{n} \prod_{k=1}^{t} d_{j_{p_k}^{(\alpha)} i_{p_k}} \\ &= \sum_{j_{p_1} j_{p_2} \cdots j_{p_t}} \prod_{k=1}^{t} d_{j_{p_k} i_{p_k}} \lambda(p_1^{j_{p_1}} p_2^{j_{p_2}} \cdots p_t^{j_{p_t}}),\end{aligned}$$

we have:

Lemma 3.1. *For any subarray T_{Ω_1} of T, two column vectors $\boldsymbol{\gamma}_{\Omega_1}$ and $\boldsymbol{\lambda}_{\Omega_1}$ are linked to each other as follows:*

$$\boldsymbol{\gamma}_{\Omega_1} = D'_{(t)}\boldsymbol{\lambda}_{\Omega_1} \text{ and } \boldsymbol{\lambda}_{\Omega_1} = \frac{1}{s^t}D_{(t)}\boldsymbol{\gamma}_{\Omega_1}, \tag{3.1}$$

where $D_{(t)}$ denotes the t-times Kronecker product of D.

Definition 3.1. The $n \times m$ array T with entries from the set of s symbols is called an orthogonal array of strength t, size n, m constraints, s symbols and index λ, if every subarray composed of t columns of T contains every possible $1 \times t$ s-ary vector with the same frequency λ. Clearly, $n = \lambda s^t$. Traditionally, such an array has been denoted as $\mathrm{OA}(n, m, s, t) : \lambda$.

Let $w_x(\boldsymbol{a}')$ be the frequency of x among the components of a vector $\boldsymbol{a}'$ and let $\boldsymbol{w}(\boldsymbol{a}')$ be the weight vector $(w_0(\boldsymbol{a}'), w_1(\boldsymbol{a}'), \cdots, w_{s-1}(\boldsymbol{a}'))$ of $\boldsymbol{a}'$.

Definition 3.2. The array T is called a balanced array of strength t, size n, m constraints, s symbols and index set $\{\mu^{(t)}_{e_0e_1\cdots e_{s-1}} | e_0 + e_1 + \cdots + e_{s-1} = t\}$, if every subarray composed of t columns of T contains every possible $1 \times t$ s-ary vector having the weight vector $\boldsymbol{w}((j_{p_1} j_{p_2} \cdots j_{p_t})) = (e_0, e_1, \cdots, e_{s-1})$ exactly $\mu^{(t)}_{e_0e_1\cdots e_{s-1}}$ times as a row of the subarray. The array is denoted as $\mathrm{BA}(n, m, s, t) : \{\mu^{(t)}_{e_0e_1\cdots e_{s-1}}\}$. Clearly,

$$n = \sum_{\sum e_r = t} \frac{t!}{e_0! e_1! \cdots e_{s-1}!} \mu^{(t)}_{e_0e_1\cdots e_{s-1}}.$$

Theorem 3.2. *In a fractional s^m factorial design T, every component $\gamma(\theta(U^1U^2 \cdots U^{s-1}))$ of the characteristic vector $\boldsymbol{\Gamma}(T)$ corresponding up to the t-factor interactions but $\gamma(\theta(\phi))$ vanishes if and only if T is an orthogonal array of strength t.*

Proof. (Necessity) From Lemma 3.1, we have

$$D'_{(t)}\boldsymbol{\lambda}_{\Omega_1} = \begin{bmatrix} \gamma(\theta(p_1^0 p_2^0 \cdots p_t^0)) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} n \\ \mathbf{0} \end{bmatrix} \text{ for every } T_{\Omega_1}.$$

Thus we have $\boldsymbol{\lambda}_{\Omega_1} = \frac{1}{s^t} D_{(t)} \begin{bmatrix} n \\ \mathbf{0} \end{bmatrix} = \frac{1}{s^t} n \boldsymbol{J}_{s^t}$, and this implies that every component $\lambda(p_1^{j_{p_1}} p_2^{j_{p_2}} \cdots p_t^{j_{p_t}}) = \frac{1}{s^t} n$ must be an integral constant λ, irrespective of the subarray T_{Ω_1}. Hereafter, $\boldsymbol{J}_x$ denotes the x-dimensional column vector whose components are all unity.

(Sufficiency) If T is an OA$(n,m,s,t):\lambda$, then we have $\boldsymbol{\lambda}_{\Omega_1}=\lambda \boldsymbol{J}_{s^t}$ for every T_{Ω_1}. Thus from (3.1) we have $\boldsymbol{\gamma}_{\Omega_1}=D'_{(t)}\boldsymbol{\lambda}_{\Omega_1}=\lambda D'_{(t)}\boldsymbol{J}_{s^t}=\begin{bmatrix}\lambda s^t\\ \mathbf{0}\end{bmatrix}$. This implies that every $\gamma(\theta(U^1U^2\cdots U^{s-1}))$ corresponding up to the t-factor interactions but $\gamma(\theta(\phi))$ vanishes. □

Theorem 3.3. *Every off-diagonal element of the information matrix $M(T)$ of a design T, i.e., $\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))$ satisfying the restriction $0<|\cup_{x\in S'}(U^x\cup V^x)|\le t$, vanishes if and only if every element of the characteristic vector $\boldsymbol{\Gamma}(T)$ corresponding up to the t-factor interactions but $\gamma(\theta(\phi))$ vanishes. The latter implies that T is an orthogonal array of strength t.*

Proof. The formula given by (2.5) shows that every one of the elements stated in the former part of the above can be expressed as a linear combination of those elements stated in the latter and satisfies the required condition. The converse is trivial. □

Theorem 3.4. *In a fractional s^m factorial design, a necessary and sufficient condition that every element $\gamma(\theta(p_1^{i_{p_1}}p_2^{i_{p_2}}\cdots p_t^{i_{p_t}}))$ of the characteristic vector $\boldsymbol{\Gamma}(T)$ corresponding up to the t-factor interactions depends on s subsets $U^x=\{p_k|i_{p_k}=x\}$ only through $|U^x|=u_x$ for $x\in S$ irrespective of the subarray indexed by a t-subset $\Omega_1=\{p_1,p_2,\cdots,p_t\}$ of Ω, or, equivalently, that every element $\gamma(\theta(U^1U^2\cdots U^{s-1}))$ of $\boldsymbol{\Gamma}(T)$ satisfying $|\cup_{x\in S'}U^x|\le t$ is invariant with respect to the symmetric group of permutation on Ω, is that T is a balanced array of strength t.*

Proof. (Sufficiency) Suppose T is a BA$(n,m,s,t):\{\mu^{(t)}_{e_0e_1\cdots e_{s-1}}\}$. Let $K_{(t)}=\|k((x_1x_2\cdots x_t),(y_0y_1\cdots y_{s-1}))\|$ be an $s^t\times\binom{s+t-1}{t}$ incidence matrix whose row indexed by an s-ary t-vector $(x_1x_2\cdots x_t)$ and column indexed by the weight vector $(y_0y_1\cdots y_{s-1})$ of some s-ary t-vector satisfying $\sum_{i=0}^{s-1}y_i=t$, such that $k((x_1x_2\cdots x_t),(y_0y_1\cdots y_{s-1}))=1$ or 0 according as $\boldsymbol{w}((x_1x_2\cdots x_t))=(y_0y_1\cdots y_{s-1})$ or not. Then from (3.1) we have:

$$\begin{aligned}
&\gamma(\theta(p_1^{i_{p_1}}p_2^{i_{p_2}}\cdots p_t^{i_{p_t}}))\\
&\quad=\sum_{\sum e_i=t}\ \sum_{j_{p_1}j_{p_2}\cdots j_{p_t}}\prod_{\ell=1}^{t}d_{j_{p_\ell}i_{p_\ell}}k((j_{p_1}j_{p_2}\cdots j_{p_t}),(e_0e_1\cdots e_{s-1}))\mu^{(t)}_{e_0e_1\cdots e_{s-1}}\\
&\quad=\sum_{\sum e_i=t}\Big\{\sum_{\mathrm{Dom}(z_x^\beta)}\prod_{x=0}^{s-1}\frac{u_x!}{z_x^0!z_x^1!\cdots z_x^{s-1}!}\prod_{\beta=0}^{s-1}(d_{\beta x})^{z_x^\beta}\Big\}\mu^{(t)}_{e_0e_1\cdots e_{s-1}},
\end{aligned}$$

where z_x^β denotes the frequency of $j_{p_k}^{(\alpha)}$'s assuming β in U^x for $x,\ \beta \in S$. Here the summation domain $\mathrm{Dom}(z_x^\beta)$ of nonnegative integers z_x^β is characterized by the following two-way restrictions:

$$\sum_{\beta=0}^{s-1} z_x^\beta = u_x \text{ for } x \in S, \text{ and } \sum_{x=0}^{s-1} z_x^\beta = e_\beta \text{ for } \beta \in S.$$

The element $\gamma(\theta(p_1^{i_{p_1}} p_2^{i_{p_2}} \cdots p_t^{i_{p_t}}))$ can, therefore, be written as $\gamma_{u_0 u_1 \cdots u_{s-1}}^{(t)}$ by indicating the cardinalities of s subsets U^x, i.e.,

$$\begin{aligned}(3.2)\qquad &\gamma_{u_0 u_1 \cdots u_{s-1}}^{(t)} \\ &= \sum_{\sum e_i = t} \Big\{ \sum_{\mathrm{Dom}(z_x^\beta)} \prod_{x=0}^{s-1} \frac{u_x!}{z_x^0! z_x^1! \cdots z_x^{s-1}!} \prod_{\beta=0}^{s-1} (d_{\beta x})^{z_x^\beta} \Big\} \mu_{e_0 e_1 \cdots e_{s-1}}^{(t)}.\end{aligned}$$

(Necessity) If $\gamma(\theta(p_1^{i_{p_1}} p_2^{i_{p_2}} \cdots p_t^{i_{p_t}}))$ depends on s subsets U^x only through their cardinalities $|U^x| = u_x,\ x \in S$, then from (3.1) we have:

$$\begin{aligned}&\lambda(\theta(p_1^{j_{p_1}} p_2^{j_{p_2}} \cdots p_t^{j_{p_t}})) \\ &\quad = \frac{1}{s^t} \sum_{\sum u_i = t} \sum_{i_{p_1} \cdots i_{p_t}} \prod_{\ell=1}^{t} d_{j_{p_\ell} i_{p_\ell}} k((i_{p_1} i_{p_2} \cdots i_{p_t}), (u_0 u_1 \cdots u_{s-1})) \gamma_{u_0 u_1 \cdots u_{s-1}}^{(t)} \\ &\quad = \frac{1}{s^t} \sum_{\sum u_i = t} \Big\{ \sum_{\mathrm{Dom}(z_x^\beta)} \prod_{\beta=0}^{s-1} \frac{e_\beta!}{z_0^\beta! z_1^\beta! \cdots z_{s-1}^\beta!} \prod_{x=0}^{s-1} (d_{\beta x})^{z_x^\beta} \Big\} \gamma_{u_0 u_1 \cdots u_{s-1}}^{(t)}.\end{aligned}$$

Here the domain $\mathrm{Dom}(z_x^\beta)$ is also characterized by the following two-way restrictions:

$$\sum_{x=0}^{s-1} z_x^\beta = e_\beta \text{ for } \beta \in S \text{ and } \sum_{\beta=0}^{s-1} z_x^\beta = u_x \text{ for } x \in S.$$

This implies T is a $\mathrm{BA}(N, m, s, t) : \{\mu_{e_0 e_1 \cdots e_{s-1}}^{(t)}\}$, where

$$\begin{aligned}(3.3)\qquad &\mu_{e_0 e_1 \cdots e_{s-1}}^{(t)} \\ &= \frac{1}{s^t} \sum_{\sum u_i = t} \Big\{ \sum_{\mathrm{Dom}(z_x^\beta)} \prod_{\beta=0}^{s-1} \frac{e_\beta!}{z_0^\beta! z_1^\beta! \cdots z_{s-1}^\beta!} \prod_{x=0}^{s-1} (d_{\beta x})^{z_x^\beta} \Big\} \gamma_{u_0 u_1 \cdots u_{s-1}}^{(t)}.\end{aligned}$$

The maximal invariant function of $(U^1 U^2 \cdots U^{s-1})$ of Ω satisfying $| \cup_{x \in S'} U^x | \le t$ with respect to the symmetric group of permutation on Ω is the set of $s-1$ nonnegative integers u_x satisfying $\sum_{x=1}^{s-1} u_x \le t$ and that of $(I^1 I^2 \cdots I^{s-1})$ is the set of $s-1$ nonnegative integers e_β satisfying $\sum_{\beta=1}^{s-1} e_\beta \le t$. The formulas (3.2) and (3.3), therefore, show that the last statement of the Theorem holds true. □

Consider, in general, an element $\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))$ of the information matrix $M(T)$ whose arguments satisfy $|\cup_{x\in S'}(U^x\cup V^x)|\le t$. Let T_{Ω_1} be a subarray composed of t columns of T satisfying $\Omega_1\supset\cup_{x\in S'}(U^x\cup V^x)$ and let U^0 and V^0 be $\Omega_1-\cup_{x\in S'}U^x$ and $\Omega_1-\cup_{y\in S'}V^y$, respectively. Let z_{xy}^{β} be the frequency of $j_{p_{xy}}^{(\alpha)}$'s assuming β in $K^{xy}=U^x\cap V^y$ for $\beta\in S$, then they satisfy the restriction $\sum_{\beta=0}^{s-1}z_{xy}^{\beta}=|K^{xy}|=k_{xy}$ for every $x,y\in S$. Suppose $\lambda^t(z_{xy}^{\beta}|x,y,\beta=0,1,\cdots,s-1)$ be the frequency of rows in the subarray in which $j_{p_{xy}}^{(\alpha)}$'s satisfy the above condition. Then, we have,

$$\begin{aligned}(3.4)\quad &\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))\\ &=\sum_{\alpha=1}^{n}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\prod_{p_{xy}\in K^{xy}}\Big(\sum_{\ell=0}^{s-1}c_{xy}^{\ell}d_{j_{p_{xy}}^{(\alpha)}\ell}\Big)\\ &=\sum_{z_{xy}^{\beta}}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\prod_{\beta=0}^{s-1}\Big(\sum_{\ell=0}^{s-1}c_{xy}^{\ell}d_{\beta\ell}\Big)^{z_{xy}^{\beta}}\lambda^t(z_{xy}^{\beta}|x,y,\beta=0,1,\cdots,s-1)\end{aligned}$$

Using (3.4), we have:

Theorem 3.5. *If the design T is composed of a balanced array of strength t with index set $\{\mu_{e_0e_1\cdots e_{s-1}}^{(t)}|e_0+e_1+\cdots+e_{s-1}=t\}$, then we have,*

$$\begin{aligned}(3.5)\quad &\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))\\ &=\sum_{\sum e_i=t}\Big\{\sum_{\mathrm{Dom}(z_{xy}^{\beta})}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\frac{k_{xy}!}{z_{xy}^0!z_{xy}^1!\cdots z_{xy}^{s-1}!}\prod_{\beta=0}^{s-1}\Big(\sum_{\ell=0}^{s-1}c_{xy}^{\ell}d_{\beta\ell}\Big)^{z_{xy}^{\beta}}\Big\}\mu_{e_0e_1\cdots e_{s-1}}^{(t)}.\end{aligned}$$

Here, the summation extends over the domain $\mathrm{Dom}(z_{xy}^{\beta})$ *of nonnegative integers z_{xy}^{β} defined by the s^2 integers $k_{xy},\ x,\ y\in S$, which are specified by the parameters $\theta(U^1U^2\cdots U^{s-1})$ and $\theta(V^1V^2\cdots V^{s-1})$ satisfying $|\cup_{x\in S'}(U^x\cup V^x)|\le t$, and by the s integers e_β, $\beta=0,1,\cdots,s-1$, specified by the index $\mu_{e_0e_1\cdots e_{s-1}}^{(t)}$ of the array as follows:*

$$\sum_{\beta=0}^{s-1}z_{xy}^{\beta}=k_{xy},\ x,\ y\in S\ \text{and}\ \sum_{x=0}^{s-1}\sum_{y=0}^{s-1}z_{xy}^{\beta}=e_\beta,\ \beta\in S.$$

The formula (3.5) shows that $\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))$ satisfying $|\cup_{x\in S'}(U^x\cup V^x)|\le t$ depends on s^2-1 nonnegative integers $u_x,\ v_y$ and $k_{xy}=|K^{xy}|$ with restriction $\sum_{x=0}^{s-1}\sum_{y=0}^{s-1}k_{xy}=t$ *for* $x,\ y\in S$ irrespective of the selected subarray T_{Ω_1}.

Consider a subarray T_{Ω_1} composed of t columns of T which covers the set $\cup_{x\in S'}(U^x \cup V^x)$ and let U^0 and V^0 be $\Omega_1 - \cup_{x\in S'}U^x$ and $\Omega_1 - \cup_{y\in S'}V^y$, respectively. From (3.4) we have,

$$\begin{aligned}
&\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))\\
&= \sum_{\alpha=1}^{n}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\prod_{p_{xy}\in K^{xy}}\Big(\sum_{\ell=0}^{s-1} c^{\ell}_{xy} d_{j^{(\alpha)}_{p_{xy}}\ell}\Big)\\
&= \sum_{\alpha=1}^{n}\Big(\sum_{\ell_{00(1)}=1}^{s-1}\cdots\sum_{\ell_{00(k_{00})}=1}^{s-1}\cdots\sum_{\ell_{xy(r)}=1}^{s-1}\cdots\sum_{\ell_{s-1s-1(k_{s-1s-1})}=1}^{s-1}\\
&\quad\Big(\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\prod_{r=1}^{k_{xy}} c^{\ell_{xy(r)}}_{xy} d_{j^{(\alpha)}_{p_{xy(r)}}\ell_{xy(r)}}\Big)\Big)\\
&= \sum_{\ell_{00(1)}=1}^{s-1}\cdots\sum_{\ell_{00(k_{00})}=1}^{s-1}\cdots\sum_{\ell_{xy(r)}=1}^{s-1}\cdots\sum_{\ell_{s-1s-1(k_{s-1s-1})}=1}^{s-1}\Big(\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\prod_{r=1}^{k_{xy}} c^{\ell_{xy(r)}}_{xy}\Big)\\
&\quad\cdot\gamma\Big(p^{\ell_{00(1)}}_{00(1)}\cdots p^{\ell_{00(k_{00})}}_{00(k_{00})}\cdots p^{\ell_{xy(r)}}_{xy(r)}\cdots p^{\ell_{s-1s-1(k_{s-1s-1})}}_{s-1s-1(k_{s-1s-1})}\Big).
\end{aligned}$$

Let $z^{\beta}_{xy}(x,y,\beta=0,1,\cdots,s-1)$ be the frequency of $\ell_{xy(r)}$'s assuming β in $K^{xy}=U^x\cap V^y$, then they satisfy the restriction $\sum_{\beta=0}^{s-1} z^{\beta}_{xy} = |K^{xy}| = k_{xy}$ for every $x,\ y\in S$.

Suppose the design T is composed of a balanced array of strength t and index set $\{\mu^{(t)}_{e_0e_1\cdots e_{s-1}} | e_0+e_1+\cdots+e_{s-1}=t\}$. Then $\gamma\Big(p^{\ell_{00(1)}}_{00(1)}\cdots p^{\ell_{00(k_{00})}}_{00(k_{00})}\cdots p^{\ell_{xy(r)}}_{xy(r)}\cdots p^{\ell_{s-1s-1(k_{s-1s-1})}}_{s-1s-1(k_{s-1s-1})}\Big)$ is equal to $\gamma^{(t)}_{u_0u_1\cdots u_{s-1}}$ irrespective of the subarray T_1 if the weight vector of $(\ell_{00(1)},\cdots,\ell_{xy(r)},\cdots,\ell_{s-1s-1(k_{s-1s-1})})$ is equal to $(u_0,u_1,\cdots,u_{s-1})$.

Thus we have another expression of (3.5), i.e.,

$$\begin{aligned}
&\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))\\
&= \sum_{\sum u_i=t}\Big\{\sum_{\mathrm{Dom}(z^{\beta}_{xy})}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\frac{k_{xy}!}{z^0_{xy}!z^1_{xy}!\cdots z^{s-1}_{xy}!}\prod_{\beta=0}^{s-1}(c^{\ell}_{xy})^{z^{\beta}_{xy}}\Big\}\gamma^{(t)}_{u_0u_1\cdots u_{s-1}},
\end{aligned}$$

where the summation extends over the domain $\mathrm{Dom}(z^{\beta}_{xy})$ of nonnegative integers z^{β}_{xy} defined by the s^2 integers k_{xy} which are specified by the parameters $\theta(U^1U^2\cdots U^{s-1})$ and $\theta(V^1V^2\cdots V^{s-1})$ satisfying $|\cup_{x\in S'}(U^x\cup V^x)|\le t$, and by the s integers u_β specified by $\gamma^{(t)}_{u_0u_1\cdots u_{s-1}}$ of the design as follows:

$$\sum_{\beta=0}^{s-1} z^{\beta}_{xy} = k_{xy},\ x,\ y\in S \text{ and } \sum_{x=0}^{s-1}\sum_{y=0}^{s-1} z^{\beta}_{xy} = u_\beta,\ \beta\in S.$$

Lemma 3.6. *The maximal invariant function of* $(U^1U^2\cdots U^{s-1})$ *and* $(V^1V^2\cdots V^{s-1})$ *with respect to the symmetric group of permutation on* Ω *is a set of* s^2-1 *nonnegative integers* u_x, v_y *and* k_{xy} *for* $1\le x,y\le s-1$ *or a set of* s^2 *nonnegative integers* k_{xy} *with* $\sum_{x=0}^{s-1}\sum_{y=0}^{s-1}k_{xy}=t$.

Combining the results of Theorem 3.4 and Lemma 3.6 we have:

Theorem 3.7. *Every element* $\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))$ *of the information matrix* $M(T)$ *whose arguments satisfy* $|\cup_{x\in S'}(U^x\cup V^x)|\le t$ *is invariant with respect to the symmetric group of permutation on* Ω *if and only if* T *is a balanced array of strength* t.

Proof. The 'if' part of this theorem is the immediate consequence of Theorem 3.5. The 'only if' part of the theorem follows from the last statement of the Theorem 3.4. □

In particular, let T in (2.2) be a design derived from a simple or full-strength s-symbol balanced array of size n and m constraints, denoted by S-BA$(n,m,s,t=m)$, having index set $\{\mu^{(m)}_{e_0e_1\cdots e_{s-1}}|e_0+e_1+\cdots+e_{s-1}=m\}$. In this case,

$$n=\sum_{e_0e_1\cdots e_{s-1}}\frac{m!}{e_0!e_1!\cdots e_{s-1}!}\mu^{(m)}_{e_0e_1\cdots e_{s-1}}.$$

$$\begin{aligned}&\varepsilon(\theta(U^1U^2\cdots U^{s-1}),\theta(V^1V^2\cdots V^{s-1}))\\&\quad=\sum_{e_0e_1\cdots e_{s-1}}\Big\{\sum_{\mathrm{Dom}(z^\beta_{xy})}\prod_{x=0}^{s-1}\prod_{y=0}^{s-1}\frac{k_{xy}!}{z^0_{xy}!z^1_{xy}!\cdots z^{s-1}_{xy}!}\prod_{\beta=0}^{s-1}\Big(\sum_{\ell=0}^{s-1}c^\ell_{xy}d_{\beta\ell}\Big)^{z^\beta_{xy}}\Big\}\mu^{(m)}_{e_0e_1\cdots e_{s-1}}.\end{aligned}$$

Under an a priori or an empirical assumption that $u+1$-factor and higher order interactions are assumed to be zero, the observation vector of the design T can be expressed as

$$\boldsymbol{y}(T)=E(u,T)\Theta(u)+\boldsymbol{e}(T)$$

in terms of the restricted design matrix $E(u,T)$, the vector $\Theta(u)$ of various effects up to u-factor interactions and the error vector $\boldsymbol{e}(T)$.

The normal equation for estimating $\Theta(u)$ is given by

$$M(u,T)\Theta(u)=E(u,T)'\boldsymbol{y}(T),$$

where, $M(u,T)=E(u,T)'E(u,T)$ is the restricted information matrix relative to $\Theta(u)$.

Theorem 3.8. *The restricted information matrix* $M(u, T)$ *of a fractional* s^m *factorial design* T *is invariant with respect to the symmetric group of permutation of* m *factors if and only if* T *is composed of an s-symbol balanced array of strength* $t = \min(m, 2u)$. *If* $2u \geq m$, *the array is necessarily simple since* $t = m$.

Proof. This is the immediate consequence of the results given in Theorems 3.4 and 3.5. □

Note that the last statement in Theorem 3.8 is a generalization of the results pointed out in Hyodo [4].

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